# Real eigenmodes of the non-Hermitian Wilson-Dirac operator and simulations of supersymmetric Yang-Mills theory

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# Super Yang-Mills theory (SYM)

**Supersymmetric Yang-Mills theory:**

$$\mathcal{L} = \mathrm{Tr}\left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \not{D} \psi - \frac{m_g}{2} \bar{\psi} \psi \right]$$

- gauge sector of supersymmetric extensions of the standard model
- $\psi$ Majorana fermion in the adjoint representation
- confinement: bound states at low energies
- symmetries: specific form of low energy effective actions

# Symmetries

SUSY ($m_g = 0$)

- supersymmetry predicts paring of bosonic and fermionic states
- no spontaneous breaking / anomaly of SUSY expected

$U_R(1)$ symmetry: $\psi \to e^{-i\theta\gamma_5}\psi$

- $U_R(1)$ anomaly: $\theta = \frac{k\pi}{N_c}$, $U_R(1) \to \mathbb{Z}_{2N_c}$
- $U_R(1)$ spontaneous breaking: $\mathbb{Z}_{2N_c} \stackrel{\langle\bar{\psi}\psi\rangle\neq 0}{\rightarrow} \mathbb{Z}_2$

# Quantized continuum SYM

- value of $\langle\bar{\psi}\psi\rangle$ is known
- exact beta function is known

Low energy effective actions:

- susy multiplets (degenerate masses)
- 1. multiplet[1]:
  mesons : $a - f_0$: $\bar{\psi}\psi$ and $a - \eta'$: $\bar{\psi}\gamma_5\psi$
  fermionic gluino-glue $(\sigma_{\mu\nu}F_{\mu\nu}\psi)$
- 2. multiplet[2]:
  glueballs: $0^{++}$ and $0^{-+}$
  fermionic gluino-glue

[1] [Veneziano, Yankielowicz, Phys.Lett.B113 (1982)]

[2] [Farrar, Gabadadze, Schwetz, Phys.Rev. D58 (1998)]

# Supersymmetric Yang-Mills theory on the lattice

**Lattice action:**

$$\mathcal{S}_L = \beta \sum_P \left(1 - \frac{1}{N_c} \Re U_P\right) + \frac{1}{2} \sum_{xy} \bar{\psi}_x \left(\mathrm{D}_w(m_g)\right)_{xy} \psi_y$$

- "brute force" discretization: Wilson fermions

$$\mathrm{D}_w = 1 - \kappa \sum_{\mu=1}^{4} \left[(1-\gamma_\mu)_{\alpha,\beta} \hat{T}_\mu + (1+\gamma_\mu)_{\alpha,\beta} \hat{T}_\mu^\dagger\right]$$

$$\hat{T}_\mu \psi(x) = V_\mu \psi(x+\hat{\mu}); \quad \kappa = \frac{1}{2(m_g+4)}$$

- links in adjoint representation: $(V_\mu)_{ab} = 2\mathrm{Tr}[U_\mu^\dagger T^a U_\mu T^b]$
  gauge group SU(2)

# Symmetries of lattice SYM

- supersymmetry always broken in local lattice theory[1]
- Wilson mass spoils mass degeneracy
- chiral symmmetry ($U_R(1)$) broken by the Wilson-Dirac operator
- no controlled breaking (Ginsparg-Wilson relation)

$\Rightarrow$ need fine tuning!

[1] [GB, JHEP 1001:024 (2010)], [Kato, Sakamoto & So, JHEP 0805:057 (2008)]

# Ward identities on the lattice

Ward identities of supersymmetry and chiral symmetry:

$$\langle \nabla_\mu J_S^\mu(x)\,\mathcal{O}(y)\rangle = m_g\langle D_S(x)\,\mathcal{O}(y)\rangle + \langle X_S(x)\,\mathcal{O}(y)\rangle$$
$$\langle \nabla_\mu J_A^\mu(x)\,\mathcal{O}(y)\rangle = m_g\langle D_A(x)\,\mathcal{O}(y)\rangle + \langle X_A(x)\,\mathcal{O}(y)\rangle + \propto \langle F\tilde{F}\,\mathcal{O}\rangle$$

- classical (tree level): $X_S(x) = O(a)$, $X_A(x) = O(a)$

renormalization, operator mixing[1,2]:

$$\langle \nabla_\mu Z_A J_A^\mu(x)\,\mathcal{O}\rangle = (m_g - \bar{m}_g)\langle D_A(x)\,\mathcal{O}\rangle + \propto \langle F\tilde{F}\,\mathcal{O}\rangle + O(a)$$
$$\langle \nabla_\mu (Z_S J_S^\mu(x) + \tilde{Z}_S \tilde{J}_S{}^\mu(x))\,\mathcal{O}\rangle = (m_g - \bar{m}_g)\langle D_S(x)\,\mathcal{O}\rangle + O(a)$$

$\Rightarrow$ tuning of $m_g$: chiral limit = SUSY limit $+O(a)$

1 [Bochicchio et al., Nucl.Phys.B262 (1985)]

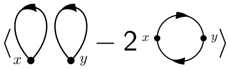
2 [Veneziano, Curci, Nucl.Phys.B292 (1987)]

# Supersymmetric chiral limit

practical problems:

- noisy signal of supersymmetric Ward identies
- chiral Ward identities contain anomaly

$$\langle \bar{\psi}(x)\gamma_5\psi(x)\,\bar{\psi}(y)\gamma_5\psi(y)\rangle =$$



define connected part as adjoint pion $(a-\pi)$

- disconnected part contains anomaly (OZI approximation)
- chiral limit: $m_{a-\pi}$ vanishes

$\Rightarrow$ possible definition of gluino mass: $\propto (m_{a-\pi})^2$

At the end the consistency with the SUSY Ward identities is checked!

# Simulations of SYM

- simulating Majorana fermions:

$$\int \mathcal{D}\psi e^{-\frac{1}{2}\int \bar{\psi} D \psi} = \mathrm{Pf}(CD) = \mathrm{sign}(\mathrm{Pf}(CD))\sqrt{\det D}$$

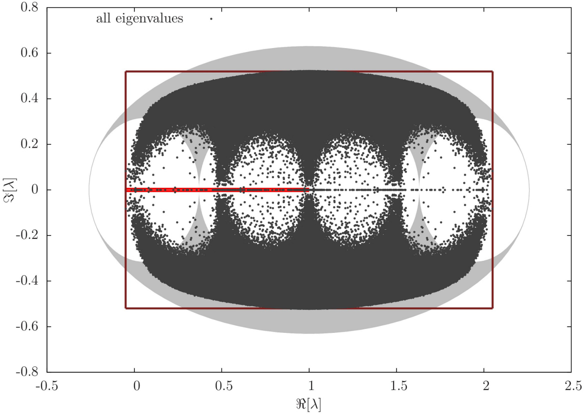
$$= \mathrm{sign}(\mathrm{Pf}(CD)) \int \mathcal{D}\bar{\phi}\mathcal{D}\phi e^{-\int \bar{\phi}(D^\dagger D)^{-1/4}\phi}$$

- reweighting with Pfaffian (Pf) sign
- PHMC algorithm: $x^{-1/4} \approx P(x)$
- improvement of the polynomial approximation: reweighting with exact contribution of smallest eigenvalues

# The sign of the Pfaffian

- $\gamma_5 \, \mathrm{D} \, \gamma_5 = D^\dagger \Rightarrow$ paring $\lambda$, $\lambda^*$
- $C \, \mathrm{D} \, C^T = \mathrm{D}^T \Rightarrow$ degenerate eigenvalues
- $|\,\mathrm{Pf}(C\,\mathrm{D})| = \sqrt{det(\mathrm{D})} = \prod_{i=1}^{N/2} |\lambda_i|$
- $|\,\mathrm{Pf}(C(\mathrm{D} - \sigma \mathbb{1}))| = \prod_{i=1}^{N/2} |\lambda_i - \sigma|$
- Pfaffian polynomial in $\sigma \Rightarrow \mathrm{Pf}(C\,\mathrm{D}) = \prod_{i=1}^{N/2} \lambda_i$
- number of negative paired real eigenvalues of D even / odd
  $\Rightarrow$ positive / negative Pfaffian
- on small lattices: checked with exact Pfaffian
- same problem (apart from degeneracy):
  determinant sign in $N_f = 1$ QCD

# Obtaining the lowest real eigenvalues of $D_w$




Focus the Arnoldi algorithm on small stripe around real axis!

# Polynomial transformation of $D_w$

- reflected spectrum: largest real eigenvalues should be computed
- Arnoldi algorithm calculates eigenvalues with real part above certain value
- computed region contains large number of unwanted eigenvalues

Two effects of transformation $D_w \to P(D_w)$:

1. focusing: better overlap of transformed wanted region with region computed by Arnoldi
2. acceleration, if eigenvalues not computed by Arnoldi compressed in a small region

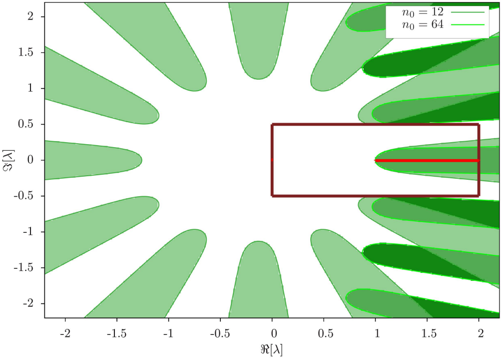
- eigenvalues of $D_w$ obtained from eigenvectors of $P(D_w)$

# Simple transformation[1] $P(\mathrm{D}_w) = (\mathrm{D}_w + \sigma_0 \mathbb{1})^{n_0}$

- complex eigenvalues "rotated away" from real axis:

$$\lambda_i = \rho_i e^{i\theta} : \quad \theta \to n_0 \theta$$

- computed regions in original spectrum:

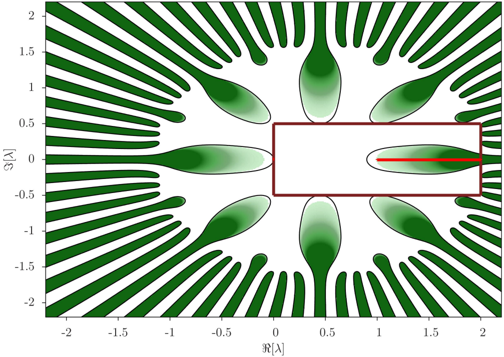



- saturation at higher orders, broad outer part of computed region

1 [H. Neff, Nucl. Phys. Proc. Suppl. **106** (2002)]

# The iterated transformation[1]

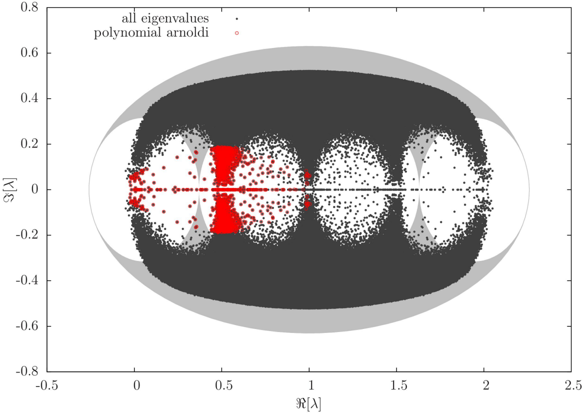
- $P(\mathrm{D}_w) = (\ldots((\mathrm{D}_w + \sigma_0 \mathbb{1})^{n_0} + \sigma_1 \mathbb{1})^{n_1} \ldots)$
  optimization at each step
- computed regions in original spectrum:




- narrow outer part of computed region!

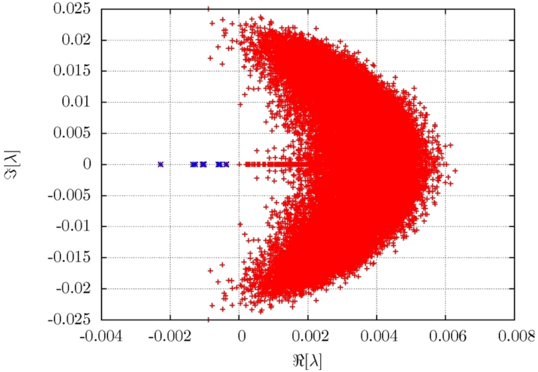
[1] [GB, Wuilloud, Comp. Phys. Comm. (2011)]

# Eigenvalues obtained from the iterated transformation




- avoids large eigenvalue densities
- increases efficiency

# Eigenvalues of $D_w$ ($32^3 \times 64$, $\kappa = 0.1495$, $\beta = 1.75$)

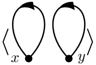



- for the determination of spectrum:
  low contribution with neg. Pfaffian (6 of 2000 configurations)
- additional acceleration: even-odd preconditioning

# Masses and particles

considered operators:

- $0^{++}$:
  - reasonable signal only with variational smearing
- fermionic gluino-glue
  - operator $\sigma^{\mu\nu}\mathrm{Tr}[\hat{F}_{\mu\nu}\psi]$
  - APE smearing on gauge fields and Jacobi smearing on $\psi$
- Meson operators $a-f_0$, $a-\eta'$:
  - disconnected contribution dominant at small gluino masses:




$$= \langle \mathrm{D}^{-1}(x,x)\,\mathrm{D}^{-1}(y,y)\rangle_{\mathrm{eff}}$$

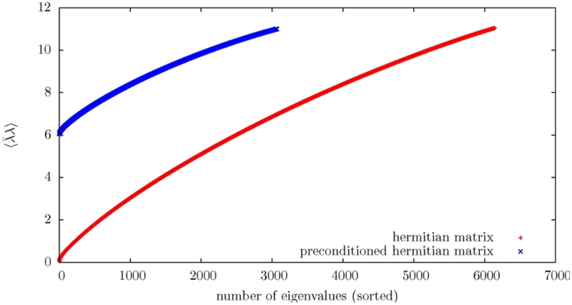
  - technique: SET (dilution, truncated solver method)
  - exact contribution of lowest $\gamma_5\,\mathrm{D}_w$ eigenvalues

# Eigenvalues of even-odd preconditioned Hermitian Wilson-Dirac operator

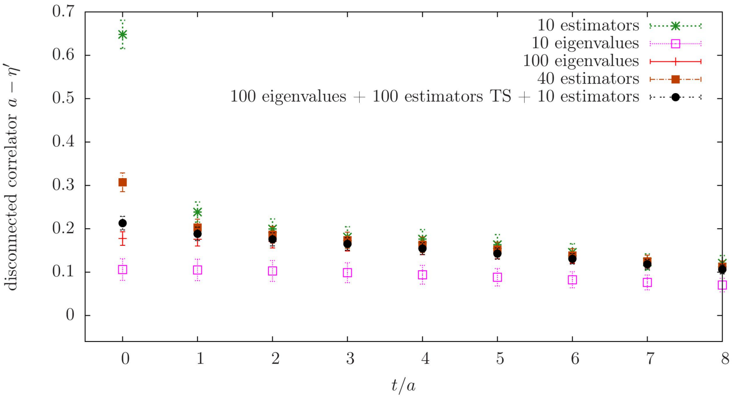
- acceleration of Arnoldi algorithm: Chebyshev polynomial

⇒ improvement: polynomial approximation of update algorithm (reweighting)

⇒ improvement: measurement of disconnected contributions and condensate



disconnected $a - \eta'$ on a $32^3 \times 64$ lattice:




- reasonable improvement at small gluino masses
- acceleration of SET inversions

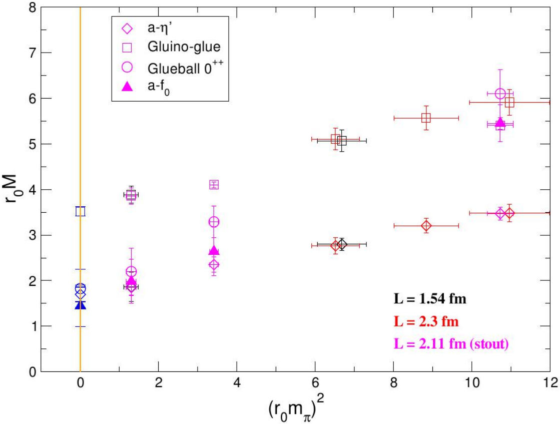
# Details of the simulations

- simulation algorithm: PHMC
- tree level Symanzik improved gauge action
- stout smearing
- Sexton-Weingarten integrator
- determinant breakup

previous simulations:

- lattice sizes: $16^3 x32$, $24^3 x48$ ($32^3 x64$)
- $r_0 \equiv 0.5\text{fm} \to a \leq 0.088\text{fm}$; $L \approx 1.5 - 2.3\text{fm}$
- $m_{a-\pi} \approx 440\text{MeV}$

# Previous SUSY Yang-Mills results




No mass degeneracy in chiral limit!
Tuning with SUSY Ward identities compatible with tuning of $m_{a-\pi}$. [Demmouche et al., Eur.Phys.J.C69 (2010)]

# New simulations at smaller lattice spacing

Before speculating about new physics: Most likely explanation are lattice artifacts!

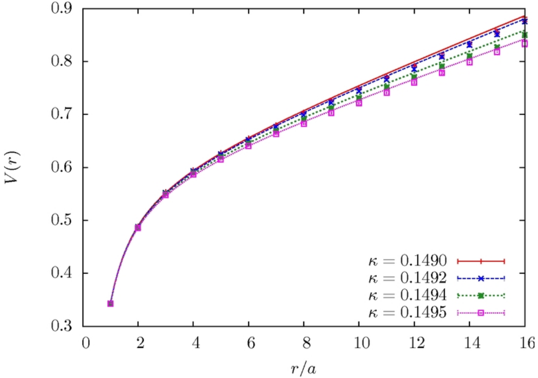
new simulations:

- volume fixed, smaller lattice spacing

$\Rightarrow$ increased $\beta$ from 1.6 to 1.75

- simulations on $32^3x64$ lattice

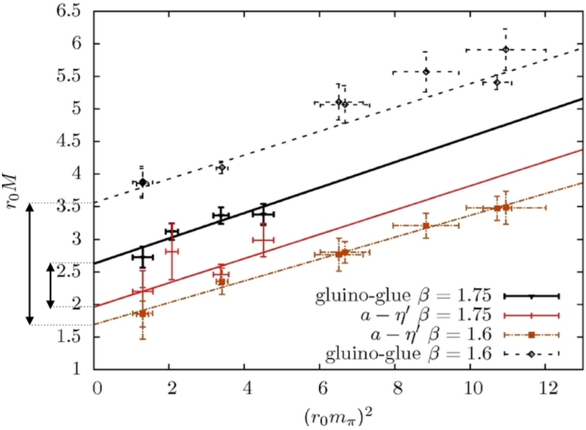
# Confinement and physical scale of the new simulations




- good agreement with $V(r) = v_0 + c/r + \sigma r$ (confining)

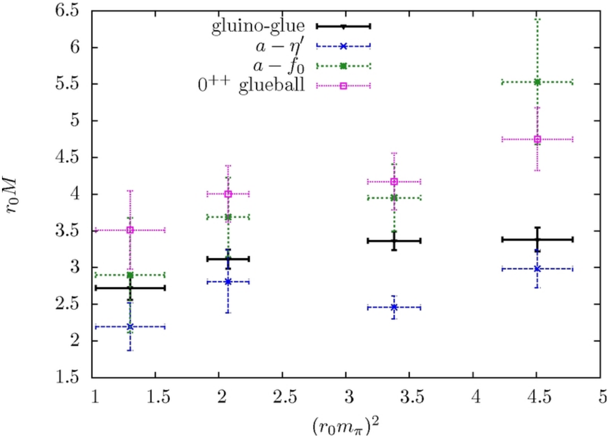
$\Rightarrow$ $a \approx 0.057\text{fm}$, $L \approx 1.8\text{fm}$

# Comparison of the mass gap between $a - \eta'$ and gluino-glue




- mass gap considerably reduced
- gluino-glue has much lower mass

# Complete spectrum obtained with the new simulations




- indicates mixing of $a - f_0$ and $0^{++}$ glueball
- in contrast to smaller lattice spacing: $a - f_0$, glueball heavier

# Conclusions

- In supersymmetric Yang-Mills theory the unavoidable breaking of SUSY on the lattice can be controlled by a fine tuning of the gluino mass ($\kappa$).
- The sign Pfaffian can be determined from the real eigenvalues of the non-Hermitian Wilson-Dirac operator.
- Polynomial acceleration of Arnoldi algorithm leads to efficient determination of lowest real eigenvalues. [1]
- The eigenvalues of the Hermitian even-odd preconditioned matrix are used to improve the algorithm and the observables.
- Possible further uses of the eigenvalue distributions ?
- The mass gap between bosonic and fermionic states is considerably reduced at a smaller lattice spacing.
- Further improvements of the action are currently investigated.

[1] Same method has been applied for the determinant sign in $N_f = 1$ QCD.