Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Lattice QCD in the heavy and dense region

Jens Langelage

ETH Zürich

in collaboration with:

Mathias Neuman, Owe Philipsen (University of Frankfurt)

Introduction

- Review strong coupling and hopping parameter expansions
- Derive analytical results for the heavy quark mass, small temperature and large chemical potential region of Lattice QCD
- Show how to include finite mass and gauge corrections
- New: $N_f = 2$ results and Isospin chemical potential

Lattice

- $(3+1)d$ hypercubic lattice with lattice spacing a
- Temporal extent N_τ
- Spatial extent $N_s \to \infty$: thermodynamic limit
- Periodic boundary conditions in time direction for bosons
- Antiperiodic boundary conditions in time direction for fermions
- Temperature: $T = \frac{1}{aN_\tau}$

Definition of the effective action

- Partition function with Wilson fermions after Grassmann integration

$$Z = \int [dU_\mu] \det[Q] \exp[S_g]$$

- Quark hopping matrix

$$\begin{aligned} Q^{ab}_{\alpha\beta,xy} &= \delta^{ab}\delta_{\alpha\beta}\delta_{xy} - \kappa \sum_{\nu=0}^{3} \Big[e^{a\mu\delta_{\nu 0}}(1+\gamma_\nu)_{\alpha\beta} U^{ab}_\nu(x)\delta_{x,y-\hat{\nu}} \\ &+ e^{-a\mu\delta_{\nu 0}}(1-\gamma_\nu)_{\alpha\beta} U^{ab}_{-\nu}(x)\delta_{x,y+\hat{\nu}} \Big] . \end{aligned}$$

- Gauge action

$$S_g = \frac{\beta}{2N_c} \sum_p \left[\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger \right] \; ,$$

- Effective theory: Integrate out spatial link variables

$$e^{S_{\mathrm{eff}}} \equiv \int [dU_k] \det[Q] \exp[S_g]$$

- Effective action depends only on Polyakov loops

$$W_i = \prod_{\tau=1}^{N_\tau} U_0(\vec{x}_i, \tau)$$

- We can use reduced Haar measure

$$\int \left[\prod_{\tau,i} dU_0(\vec{x}_i, \tau)\right] \quad \to \quad \int \left[\prod_i dW_i\right] \to \int \left[\prod_i dL_i\right] e^{V(L,L^*)} \,,$$

$$V(L, L^*) \quad = \quad \frac{1}{2} \ln \left(27 - 18|L|^2 + 8\mathrm{Re}L^3 - |L|^4\right) .$$

- Dimensionally reduced theory formulated in complex numbers

Remarks

- Effective theory not known in full parameter space $(\beta, \kappa_f, N_\tau, \mu)$
- Need infinite tower of interaction terms
- So far we do not now the full dependence for even some of the effective couplings
- Approximate effective couplings in a strong coupling and hopping parameter expansion around $(\beta, \kappa) = (0, 0)$, where the quarks are infinitely heavy and the lattice is maximally coarse
- There is still a sign problem, albeit milder the heavier the quarks

Pure gauge theory

- Expand Wilson action in characters

$$\exp\left[\frac{\beta}{6}\left(\mathrm{tr}\,U + \mathrm{tr}\,U^\dagger\right)\right] = c_0(\beta)\left[1 + \sum_{r\neq 0} d_r a_r(\beta)\chi_r(U)\right]$$

- Express results in fundamental expansion coefficient $u = a_f(\beta)$
- Leading order corresponds to a chain of N_τ plaquettes

$$e^{S^g_{\mathrm{eff}}} = u^{N_\tau}\sum_{<ij>}\mathrm{Tr}\,W_i\,\mathrm{Tr}\,W_j$$

- Need group integral

$$\int dU U_{ij} U^\dagger_{kl} = \frac{1}{3}\delta_{il}\delta_{jk}$$

Relevance for small temperatures

- Temperature on the lattice

$$T = \frac{1}{N_\tau a}$$

- Cold means large a ($=$ small β) or large N_τ
- Continuum physics then needs large N_τ, where u^{N_τ} is exponentially small, since $u < 1$
- Corrections increase the effective coupling, but it remains vanishingly small
- Next-to-nearest neighbour interactions are even stronger suppressed
- For now, neglect gauge corrections $=$ Strong coupling limit

Hopping parameter expansion

Static quark determinant

- Write the hopping matrix as

$$Q = 1 - \sum_{\pm\nu} \kappa_\nu (1+\gamma_\nu) U_\nu(x) = 1 - T - S$$

- Static limit: Neglect spatial quark hops. Determinant can be evaluated exactly

$$\det\left[Q_{\mathrm{stat}}\right] \equiv \det\left[1 - T^+\right]\left[1 - T^-\right]$$

$$= \prod_{f,i} \det\left[1 + c_f W_i\right]^2 \det\left[1 + \bar{c}_f W_i^\dagger\right]^2$$

- Polyakov loops thus have a fugacity factor

$$c \equiv \left(2\kappa e^{a\mu}\right)^{N_\tau} = (2\kappa)^{N_\tau} e^{\frac{\mu}{T}} = \exp\left[\frac{\mu - \frac{1}{3} m_B^{\mathrm{stat}}}{T}\right]$$

$N_f = 1,\ T = 0$

- Partition function: Single site problem

$$Z = \left\{ \int dW \det\left[1 + cW_i\right]^2 \det\left[1 + \bar{c}W_i^\dagger\right]^2 \right\}^{N_s^3}$$

- Pressure (omitting $\bar{c}$ contributions)

$$P = \frac{T}{V} \ln Z \quad = \quad \frac{1}{N_\tau a^4} \ln\left[1 + 4c^3 + c^6\right]$$

- Number density

$$a^3 n_q = \frac{12c^3 + 6c^6}{1 + 4c^3 + c^6}$$

$N_f = 1, \ T = 0$

- Number density

$$a^3 n_q = \frac{12c^3 + 6c^6}{1 + 4c^3 + c^6}$$

- Limit $T \to 0$

$$\lim_{T\to 0} c = \lim_{T\to 0} \exp\left[\frac{\mu_q - \frac{1}{3} m_B^{\mathrm{stat}}}{T}\right]$$

- $\to$ Onset at $\mu_q = \frac{1}{3} m_B^{\mathrm{stat}}$
- Number density at $T = 0$

$$\lim_{T\to 0} a^3 n_q \quad = \begin{cases} 0, & \mu_q < \frac{1}{3} m_B^{\mathrm{stat}} \\ 2N_c, & \mu_q > \frac{1}{3} m_B^{\mathrm{stat}} \end{cases}$$

Pauli principle

- Consider leading c behavior

$$Z = \left\{ \int dW \exp\left[2cW_i\right] \right\}^{N_s^3}$$

- Group integral now harder, but has been solved
- Result: No saturation for this truncated version of the effective theory, violation of the Pauli principle
- Lesson: Need to each order in κ the complete dependence on chemical potential encoded in c

$N_f = 2,\ T = 0,\ \mu_u = \mu_d$

- Partition function: Omitting $\bar{c}$ contributions

$$
\begin{aligned}
Z &= \left\{ \int dW \prod_{f=1}^{2} \det\left[1 + c_f W_i\right]^2 \det\left[1 + \bar{c}_f W_i^\dagger\right]^2 \right\}^{N_s^3} \\
&= 1 + 4c_u^3 + 4c_d^3 + 6c_u^2 c_d + 6c_u c_d^2 + \ldots + c_u^6 c_d^6
\end{aligned}
$$

- Onset at (assuming $m_u < m_d$)

$$\mu_q^c = \frac{1}{6}\left(m_{uuu}^{\mathrm{stat}} + m_{ddd}^{\mathrm{stat}}\right)$$

- Number density at $T = 0$

$$\lim_{T\to 0} a^3 n_q = \begin{cases} 0, & \mu_q < \frac{1}{6}\left(m_{uuu}^{\mathrm{stat}} + m_{ddd}^{\mathrm{stat}}\right) \\ 2N_c N_f, & \mu_q > \frac{1}{6}\left(m_{uuu}^{\mathrm{stat}} + m_{ddd}^{\mathrm{stat}}\right) \end{cases}$$

$N_f = 2,\ T = 0,\ \mu_u = -\mu_d$

- Partition function: Omitting $\bar{c}_u$ and c_d contributions

$$
\begin{aligned}
Z &= \left\{ \int dW \prod_{f=1}^{2} \det\left[1 + c_f W_i\right]^2 \det\left[1 + \bar{c}_f W_i^\dagger\right]^2 \right\}^{N_s^3} \\
&= 1 + 4 c_u \bar{c}_d + \ldots + c_u^6 \bar{c}_d^6
\end{aligned}
$$

- Onset at (assuming $m_u < m_d$)

$$\mu_q^c = \frac{1}{2} m_{u\bar{d}}^{\mathrm{stat}} = \frac{1}{2} m^{\mathrm{stat}}(\pi^+)$$

- Number density

$$
\lim_{T \to 0} a^3 n_q = \begin{cases} 0, & \mu_q < \frac{1}{2} m_{u\bar{d}}^{\mathrm{stat}} \\ 2 N_c N_f, & \mu_q > \frac{1}{2} m_{u\bar{d}}^{\mathrm{stat}} \end{cases}
$$

Corrections

Leading fermionic correction

- Introduce some definitions and perform some algebra

$$\begin{aligned}\det[Q] &= \det[1-T-S] = \det[1-T][1-(1-T)^{-1}S] \\ &= \det[Q_{\rm stat}][Q_{\rm kin}]\end{aligned}$$

- Split S in positive and negative directions

$$\begin{aligned}\det[Q_{\rm kin}] &= \det[1-(1-T)^{-1}(S^+ + S^-)] = \det[1-P-M] \\ &= \exp\left[\mathrm{Tr}\ln(1-P-M)\right]\end{aligned}$$

- Trace in coordinate space (=closed loops), i.e. only terms with an equal number of P and M survive
- To $\mathcal{O}(\kappa^2)$

$$\det[Q_{\rm kin}] = \det[1-PM][1+\mathcal{O}(\kappa^4)]$$

- Higher corrections can be systematically included in this way

Static propagator

- Essential ingredient for computing corrections:

$$D = (1 - T)^{-1} = (1 - T^{+})^{-1} + (1 - T^{-})^{-1} - 1$$

- Only temporal hops are involved: closed form expression available (but it is a quite lengthy expression). Splitting it in spin space:

$$D_{t_1,t_2} = A_{t_1,t_2} + \gamma_0 B_{t_1,t_2}$$

- For the leading correction we only need the part of B_{t_1,t_2} diagonal in time

$$B_{t_1,t_1} = -\frac{1}{2}\frac{cW}{1+cW} + \frac{1}{2}\frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}$$

Static propagator: $z = (2\kappa e^{a\mu})$

$$
\begin{aligned}
A &= \delta_{\tau_1\tau_2}\left(1 - \frac{1}{2}\frac{cW}{1+cW} - \frac{1}{2}\frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}\right) \\
&+\Theta(\tau_2-\tau_1)\frac{1}{2}\left(\frac{z^{\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW} - \frac{\bar{z}^{N_\tau+\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right) \\
&+\Theta(\tau_1-\tau_2)\frac{1}{2}\left(-\frac{z^{N_\tau+\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW} + \frac{\bar{z}^{\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right) \\
B &= \delta_{\tau_1\tau_2}\frac{1}{2}\left(-\frac{cW}{1+cW} + \frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}\right) \\
&+\Theta(\tau_2-\tau_1)\frac{1}{2}\left(\frac{z^{\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW} + \frac{\bar{z}^{N_\tau+\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right) \\
&+\Theta(\tau_1-\tau_2)\frac{1}{2}\left(-\frac{z^{N_\tau+\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW} - \frac{\bar{z}^{\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right)
\end{aligned}
$$

Static propagator

- Essential ingredient for computing corrections:

$$D = (1 - T)^{-1} = (1 - T^{+})^{-1} + (1 - T^{-})^{-1} - 1$$

- Only temporal hops are involved: closed form expression available (but it is a quite lengthy expression). Splitting it in spin space:

$$D_{t_1,t_2} = A_{t_1,t_2} + \gamma_0 B_{t_1,t_2}$$

- For the leading correction we only need the part of B_{t_1,t_2} diagonal in time

$$B_{t_1,t_1} = -\frac{1}{2}\frac{cW}{1+cW} + \frac{1}{2}\frac{\bar{c}W^{\dagger}}{1+\bar{c}W^{\dagger}}$$

Result for leading correction

- Now the spatial link integration is nontrivial

$$\int [dU_k] \det[1 - PM] = \int [dU_k][1 - \mathrm{Tr}PM + \mathcal{O}(\kappa^4)]$$

- For every spatial link there are now exactly one U and one $U^\dagger$ and with

$$\int dU \; U_{ij} U^\dagger_{kl} = \frac{1}{N_c} \delta_{il} \delta_{jk}$$

we have (omitting $\bar{c}$ contributions)

$$\int [dU_k][1 - \mathrm{Tr}PM] = \prod_{<ij>} \left[1 - \frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

Number density?

- Partition function for N_f flavors

$$Z = \int [dW] \prod_{f,i} \det[1 + c_f W_i] \prod_{f,<ij>} \left[1 - \frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{c_f W_i}{1 + c_f W_i} \mathrm{Tr} \frac{c_f W_j}{1 + c_f W_j}\right]$$

- No analytic solution, simulation via Complex Langevin Dynamics
- Taylor expansion in $\frac{\kappa^2 N_\tau}{N_c}$ leads again to the same step function at $T = 0$
- We can not see changes in onset due to interactions by inspecting a Taylor expansion to a finite order

Higher order corrections

- Nearest-neighbour interaction also contains higher orders

$$\det[1 - PM] = \exp\left[-\mathrm{Tr}\sum_n \frac{1}{n}(PM)^n\right]$$

- Full kinetic determinant contains couplings between all points

$$\begin{aligned}\det[Q_{\mathrm{kin}}] &= \det[1 - PM - P^2M^2 - \ldots] \\ &= \det[1 - PM][1 - P^2M^2][1 + \mathcal{O}(\kappa^6)]\end{aligned}$$

- Note that $P = \sum_{i=1}^{3} P_i$, i.e. P^2 mixes different directions
- Up to now: κ^4 contribution is finished, for certain situtations (large N_τ behavior) even much higher orders are computable

Gauge corrections

- Leaving the strong coupling limit we have to compute

$$\int [dU_k] \det[Q_{\mathrm{kin}}] \prod_p \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right]$$

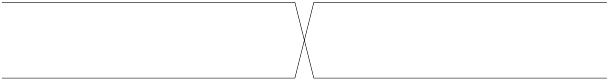
- Now: double series expansion in β and κ
- No conceptual difficulties, only a larger number of terms to be computed
- So far: results up to and including $\mathcal{O}(\kappa^n \beta^m)$ with $n + m = 4$

Resummations

- Why resummations:
  - Include higher order graphs at minimal cost
  - Does not compromise correctness of originial series
  - But: At some point, imperative to include them
- Leading κ correction

$$\int [dU_k] \det[1 - PM] = \prod_{<ij>} \left[1 - \frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

- Small T region: coefficient diverges for fixed κ



Resummations

- These terms exponentiate

- After resumming these terms

$$\int [dU_k] \det[1 - PM] = \prod_{<ij>} \exp\left[-\frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

- Similar results hold also for other terms, e.g. the single Polyakov line coupling receives renormalization

Conclusions

- Hopping expansion straightforward, but correct physics requires some attention (Pauli principle)
- Resummations necessary to improve convergence (or even to get meaningful results)
- Probe thermal lattice QCD with (heavy) quarks at small temperatures
- Some interesting results even with analytical methods